Finnish High School Mathematics Contest
School Year 2014 – 2015

The Finnish High School Mathematics Contest is organized annually by MAOL, the Finnish Association of Mathematics and Science Teachers. The Finnish Mathematical Society's Training Section participates in problem selection and marking of solutions. The contest is the first stage in the Finnish IMO Team selection process. It takes place in two rounds. As the division into grades no more exists in the Finnish high school system, Round One has two divisions in which the age of the contestants is limited, and one division open for all students regardless of their age or school level. Basic Division roughly caters for students in their first high school year and Intermediate Division for those on their second year. The exams in the various divisions may have problems in common.

Round One is organized in schools in November. Altogether about 1500 students participate, the number rather evenly divided between the divisions. Time allowed is 120 minutes. Following the current school praxis, calculating machines and the use of a canonical formula collection have been permitted. Round Two is in Helsinki in January or February. About 20 best students from all divisions in Round One are invited, most of them from the Open Division. In Round Two, the working time is 180 minutes, and no calculating aids are permitted.

In Round One, Basic and Intermediate Division, part of the problems are multiple choice. The number of correct answers to these varies, unlike in many other competitions, it is not restricted to one for each problem.

Problems

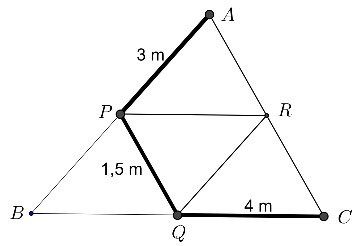
First Round, November 11, 2014

Basic Level

1. A train goes from Duckburg to Goose Mountain. The stops take 5 % of the travel time. One wants to shorten the travel time by 10 %, but the time for the stops cannot be changed. The speed of the train has to be increased by ("approximately" means one percentage point accuracy)

a) 10 % b) less than 15 % c) approximately 12 % d) approximately 15 %

2. A triangular frame is constructed using metal bars. The frame is strengthened by bars connecting the midpoints of the sides of the triangles. In the picture, which may not be accurate in measure, some lengths of bars or their parts have been indicated by bolder segments. How much metal bar has been used in the construction?



a) at least 24 m b) at least 25 m c) at least 26 m d) at least 27 m

3. Do there exist positive real numbers a and b such that

a) $2 : \left(\frac{1}{a} + \frac{1}{b}\right) \geq \sqrt{ab}$ b) $2 : \left(\frac{1}{a} + \frac{1}{b}\right) > \sqrt{ab}$

c) $2 : \left(\frac{1}{a} + \frac{1}{b}\right) = \sqrt{ab}$ d) $2 : \left(\frac{1}{a} + \frac{1}{b}\right) < \sqrt{ab}$

4. Consider the natural number

$$\begin{aligned} N = \ & 97\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531 \\ & 097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531 \\ & 097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531\,097\,531 \end{aligned}$$

in which all the odd digits appear in descending order 25 times with zeroes in between. This number N is divisible by the integer

a) 9 b) 5 c) 3 d) 11

5. In Finland, coins of denominations 5, 10, 20 and 50 cents are in use. In how many ways can one pay a bill of one euro using these coins?

a) less than 45 b) less than 50 c) 50 d) more than 50

6. The polynomial $(3x-1)^7$ can be expanded in the form

$$a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Then $a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1$ is

a) odd b) -115 c) 129 d) 0

7. Compute the lengths of the diagonals of a regular octagon inscribed in a circle of radius r.

8. Let n be a non-negative integer. In how many ways can the persons A, B and C divide n similar candies between themselves?

Intermediate Division

1. Basic Division, Problem 4.

2. Basic Division, Problem 5.

3. Basic Division, Problem 6.

4. Compute the lengths of the diagonals of a regular 12-gon inscribed in a circle of radius r.

5. We know that the real numbers x and y satisfy $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right) = 1$. Determine the possible values of $x+y$.

6. In the city of Foolville services are produced using the so called orderer-provider model.[1] The director of the office for ordering welfare services in traffic demands that the director of the office for providing welfare services in traffic should organize the bus lines in Foolville in such a way that each line has 4 stops, one can travel from each stop to any other without changing the line in between and no pair of stops is served by more than one line. After a week, the director presents two alternative models, containing a different number of lines, that comply with the given requirements. How many lines do the models consist of?

[1] The story is not entirely fictitious.

Open Division

1. Intermediate Division, Problem 5.

2. We assume that an aircraft engine will break down during a flight with probability p and that the breakdown of an engine is independent of what happens to the other engines of the aircraft. We know that a two engine plane can fly with a single engine and a four engine plane can fly, if there is one working engine on both sides of the plane. For which values of p a two engine plane is safer than a four engine plane?

3. Consider an equilateral triangle ABC. Let P be an arbitrary point on the shorter arc AC of the circumcircle of ABC. Show that $|PB| = |PA| + |PC|$.

4. Laura and Risto play the following game: There are $\ell \geq 2$ plates on the table, all empty at the beginning. Laura starts each round by moving some of the plates to her left and the rest to her right side. Risto then chooses the plates on either side and puts a raisin on each of these plates; he also empties the other plates. Laura can end the game at this point and win all the raisins on one plate, or else start a new round. Prove that if Risto plays in the best way possible, then Laura can win at most $\ell - 1$ raisins.

Round Two, January 30, 2015

1. Solve the equation

$$\sqrt{1+\sqrt{1+x}} = \sqrt[3]{x}$$

for $x \geq 0$.

2. The lateral edges of a right square pyramid are of length a. Let $ABCD$ be the base of the pyramid, E its top vertex and F the midpoint of CE. Assuming that BDF is an equilateral triangle, compute the volume of the pyramid.

3. Determine the largest integer k for which 12^k is a factor of $120!$.

4. Let n be a positive integer. Every square in a $n \times n$-square grid is either white or black. How many such colourings exist, if every 2×2-square consists of exactly two white and two black squares? The squares in the grid are identified as e.g. in a chessboard, so in general colourings obtained from each other by rotation are different.

5. Mikko takes a multiple choice test with ten questions. His only goal is to pass the test, and this requires seven points. A correct answer is worth one point, and answering wrong results in the deduction of one point. Mikko knows for sure that he knows the correct answer in the six first questions. For the rest, he estimates that he can give the correct answer to each problem with probability p, $0 < p < 1$. How many questions Mikko should try?

Answers and Solutions

Basic Division

1. Denote the distance of the stations by s, the original train speed by v_0, the new train speed by v_1, and the original total travel time by T_0. Then

$$v_0 = \frac{s}{0.95 \cdot T_0}$$

and

$$0.9 \cdot T_0 = \frac{s}{v_1} + 0.05 \cdot T_0$$

or

$$v_1 = \frac{s}{0.85 \cdot T_0}.$$

So the asked percentage can be computed from

$$\frac{v_1 - v_0}{v_0} = \frac{\frac{1}{0.85} - \frac{1}{0.95}}{\frac{1}{0.95}} = \frac{0.1}{0.85} \approx 0.118.$$

So b) and c) are correct, a) and d) false.

2. The length of a bar joining the midpoints of two sides is exactly one half of the third side. So the total length of the bars is three times the length of the bars marked in the picture, i.e. $3 \cdot 8.5$ m $= 25.5$ m. a) and b) are correct, c) and d) false.

3. Setting $a = b$ one immediately sees that a) and c) are correct. That d) also is true can be seen e.g. by setting $a = 1$ and $b = 4$. Assuming b), one arrives at the contradiction $0 > (a-b)^2$ for all a, b. So b) is false. [A realistic assumption is that the average contestant is unaware of the basic facts of inequalities.]

4. None of the alternatives is correct: the number ends in a 1, so it is not divisible by 5, its digit sum is $25 \cdot (9 + 7 + 5 + 3 + 1) = 625$ which is not divisible by 9 or 3, and the alternating digit sum is $25 \cdot (9 - 7 + 5 - 3 + 1) = 5^3$, which is not a multiple of 11.

5. The computation may be simplified, if we count the number of different ways to pay a $10 \cdot n$ cent bill, where n is an integer. There is one way to pay the bill using 5 c coins only. With 5 and 10 cent coins, the number of possibilities is $n+1$: one can use k 10 c coins, $0 \le k \le n$, and $2(n-k)$ 5 c coins. If 20 c coins are also in use, one can use k of them, where $20 \cdot k \le 10 \cdot n$, or $k \le \left\lfloor \frac{n}{2} \right\rfloor$, and for any choice of k have $n - 2k + 1$ possibilities of paying the remaining $10 \cdot n - 20 \cdot k = 10 \cdot (n-2k)$ cents with 5 c and 10 c coins. Summing the arithmetic sequence we see that the total number of possibilities is

$$\sum_{k=0}^{\lfloor} n/2\rfloor (n-2k+1) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \cdot \frac{1}{2}\left((n+1) + \left(n - \left\lfloor \frac{n}{2} \right\rfloor + 1\right)\right)$$

$$= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(n + 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil + 1\right).$$

In the problem, $n = 10$. If no 50 c coins are used, the number of possibilities is $(5+1)(5+1) = 36$. If just one 50 c coin is used, the number of possibilities is

$$\left(\left\lfloor \frac{5}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{5}{2} \right\rceil + 1\right) = 3 \cdot 3 = 12.$$

Finally, there is just one way of paying with two 50 c coins. The total number of possibilities is $36 + 12 + 1 = 49$, so b) is correct, a), c) and d) false.

6. $P(0) = a_0 = (-1)^7 = -1$ and

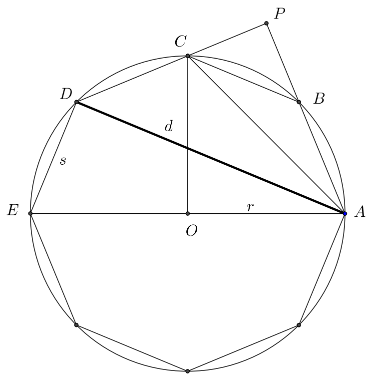
$$P(1) = \sum_{k=0}^{7} a_k = 2^7 = 128.$$

So

$$\sum_{k=1}^{7} a_k = 128 - (-1) = 129.$$

c) is correct. a), b) and d) false.

7. A regular octagon $ABCDE\ldots$ has diagonals of three sizes, e.g. AC, AD and AE. Letting O be the center of the circumcircle of the octagon, the isosceles right triangle ACO immediately gives $AC = \sqrt{2}r$, and of course $AE = 2r$. To compute $d = AD$, we denote the side of the octagon by s and extend AB and DC to meet at P. Then BPC and APD are isosceles right triangles, and so $BP = \dfrac{1}{\sqrt{2}}s$ and



$$d = \sqrt{2}\left(s + \frac{1}{\sqrt{2}}s\right) = (\sqrt{2}+1)s.$$

By Thales, ADE is a right triangle, so $d^2 + s^2 = 4r^2$. Plugging s in terms of d here and simplifying, one gets $d = r\sqrt{2+\sqrt{2}}$.

8. The person administering the division of the candies can put all of them in a row and then utilize two division marks which she inserts in the row. She then gives all the candies left of the left division mark to A, candies between the marks to B and the rest to C. Different ways to place the division marks lead to different ways of dividing the candies. There are $n+2$ items in the row, and any set of two items defines the division marks. So the number of different divisions is

$$\binom{n+2}{2} = \frac{1}{2}(n^2+3n+1).$$

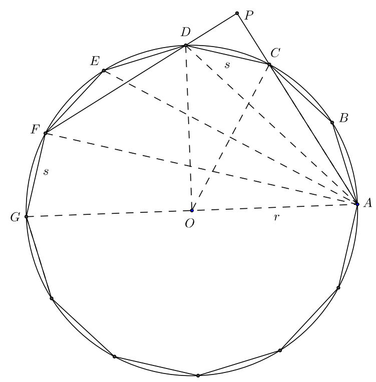
Intermediate Division

1. See Basic Division, Problem 4.

2. See Basic Division, Problem 5.

3. See Basic Division, Problem 6.

4. Let $ABCDEFG\ldots$ be a regular 12-gon, s its side length and O its circumcentre. The polygon has diagonals of five different lengths, e.g. AC, AD, AE, AF and AG. The equilateral triangle OAC yields $AC = r$ and the right isosceles triangle ADO gives $AD = r\sqrt{2}$.Clearly, $AG = 2r$ and the right triangle AEG with $AE = r$ gives $AD = r\sqrt{3}$. To compute AF, we extend AC and FD to meet at P and notice that $\angle FAC = \angle DFA = 3 \cdot 15° = 45°$. So APF as well as CPD are isosceles right triangles, which makes $s = \sqrt{2} \cdot PC$ and $AF = \sqrt{2} \cdot (AC + CP)$. So



$$AF = s \cdot \frac{AC+CP}{CP} = s \cdot \frac{r + \frac{s}{\sqrt{2}}}{\frac{s}{\sqrt{2}}} = r\sqrt{2} + s.$$

The right triangle AFG then gives $AF^2 + s^2 = 4r^2$ or $s^2 + \sqrt{2} \cdot rs - r^2 = 0$. Solving the equation in s we get

$$s = \left(\frac{\sqrt{6}}{2} - \frac{1}{\sqrt{2}}\right) r$$

which gives

$$AF = \frac{1+\sqrt{3}}{\sqrt{2}} r.$$

(It is easy to see that we can also write for instance $AF = \sqrt{2+\sqrt{3}} \cdot r$.)

5. The numbers x and y satisfy

$$x+\sqrt{x^2+1}=\frac{1}{y+\sqrt{y^2+1}}=-y+\sqrt{y^2+1}$$

or

$$x+y=\sqrt{y^2+1}-\sqrt{x^2+1}.$$

But since x and y are interchangeable in the equation given in the problem, we also have

$$x+y=\sqrt{x^2+1}-\sqrt{y^2+1}.$$

So $x+y=-(x+y)$, and $x+y=0$.

6. Let $\mathcal{P}$ be se set all bus stops. We can view any bus line L as a subset of $\mathcal{P}$. Consider a stop P. If $P\in L$ and $P\in L'$, then $L\cap L'=\{P\}$. For any $Q\in\mathcal{P}$ there is a line L such that $\{P, Q\}\subset L$. We see that the lines passing through P make up a partition of $\mathcal{P}\setminus\{P\}$. Since every line consists of four stops, $\mathcal{P}$ consists of $1+3k$ stops, for some k. Now it is clear that $k=1$, a single line with four stops, provides one system satisfying the requirements. Let us see what happens for $k\geq 2$. Then there are at least two lines L and L' passing through P. Moreover, there are stops $Q\in L$, $Q'\in L'$, $Q\neq Q'$. Also, there is a line L'' such that $\{Q, Q'\}\subset L''$. There are altogether four stops in L'' and each of them is connected to P. Since any line through P must have a common stop with L'', there can only be four lines passing through P. i.e. $k=4$. Counting the lines by the stops on L, we see that the number of lines has to be $1+4\cdot 3=13$.

The matrix

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	*	*	*	*									
2	*				*	*	*						
3	*							*	*	*			
4	*										*	*	*
5		*			*			*			*		
6		*				*			*			*	
7		*					*			*			*
8			*		*				*				*
9			*			*				*	*		
10			*				*	*				*	
11				*	*					*		*	
12				*		*		*					*
13				*			*		*		*		

where the rows stand for lines and columns for stops shows that it is indeed possible to make this system of 13 lines and 13 stops.

Open Division

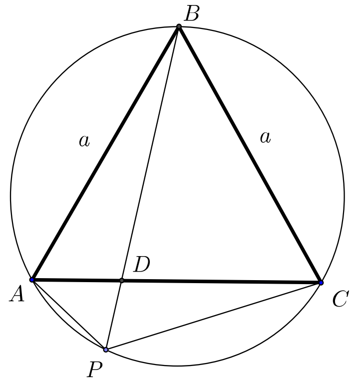
1. See Intermediate Division, Problem 5.

2. A four engine plane can be considered as consisting of two twin engine planes. Assuming the probability that a two engine plane works to be q, the probability that a four engine plane works is q^2. Since $0 < q < 1$, $q^2 < q$.

3. Let AC meet BP at D. Then $\angle APB = \angle ABC = 60° = \angle CAB$. The triangles APB and DAB are similar, as are CPB and DCB (two pairs of equal angles). Set $|AC| = |BC| = |AB| = a$. The similarities imply

$$|AP| = |AD|\frac{|BP|}{a}, \quad |PC| = |DC|\frac{|BP|}{a}. \qquad (1)$$

Because $|AD| + |DC| = a$, adding the equations (1) gives the claim immediately.



4. Denote by a_j the number of plates having at least j raisins, for any $j \in \mathbb{N}$. We show that Risto can always play in such a way that after his move (i.e. when it is Laura's move) $a_j \le \ell - j$, for all $j \le \ell$ and $a_j = 0$, for $j > \ell$. We prove this by induction. When the game starts, all plates are empty which means that $a_0 = \ell = \ell - 0$ and $a_j = 0 \le \ell - j$ for $j \le \ell$. Assume that after some move by Risto $a_j \le \ell - j$, $j \le \ell$, and $a_j = 0$, $j > \ell$. Of the a_j plates with at least j raisins, Laura moves b_j to her left and c_j to her right side. Let r be the largest number of raisins on any plate. We know that $r < \ell$. We can assume that at least one of the plates with r raisins is on the left hand side of Laura. So $b_r > 0$. Now, as his move, Risto empties all plates to the left of Laura and add a raisin on every plate to her right. This means that there are now c_{j-1} plates with at least j raisins. But the induction assumption implies $c_{j-1} = a_{j-1} - b_{j-1} \le a_{j-1} - b_r < a_{j-1} \le \ell - (j-1) = \ell - j + 1$. As one of the inequalities is strict, $c_{j-1} \le \ell - j$. Of course $j = 0$ also satisfies the induction claim. Now the number of plates with at least ℓ raisins is $\ell - \ell = 0$. So no plate can have more than $\ell - 1$ raisins.

Final round

1. Set $y = \sqrt{1+x}$. Then $y \ge 1$ and $x = y^2 - 1$. The equation to be solved is $\sqrt{1+y} = \sqrt[3]{y^2-1}$. Raising both sides to power 6, we get

$$(1+y)^3 = (y^2-1)^2 = (1+y)^2(1-y)^2$$

and

$$1 + y = (1-y)^2 = 1 - 2y + y^2, \qquad y^2 = 3y.$$

The only possibility is $y = 3$ or $x = 3^2 - 1 = 8$. That $x = 8$ indeed is a solution, can be checked with the original equation.

2. Let E' be the centre of the square $ABCD$. Because $ABCDE$ is a right pyramid, EE' is the altitude. If F' the point of the segment AC which satisfies $FF' \perp AC$, then F' is the midpoint of $E'C$. From the similar triangles $EE'C$ and $FF'C$ we see that $EE' = 2 \cdot FF'$. The diagonal BD of $ABCD$ measures $a\sqrt{2}$. Since E' is the midpoint of BD and BDF is an equilateral triangle, FE' is the altitude of BDF and $FE' = \dfrac{\sqrt{6}}{2}a$.

The triangle $FE'F'$ is a right triangle and $E'F' = \dfrac{1}{4}AC = \dfrac{\sqrt{2}}{4}a$. By Pythagoras,

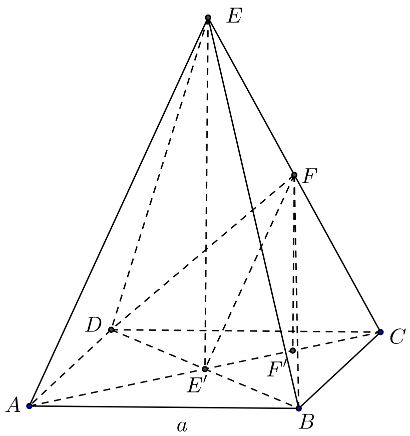
$$FF'^2 = FE'^2 - E'F'^2 = \left(\frac{6}{4} - \frac{1}{8}\right)a^2 = \frac{11}{8}a^2.$$

So

$$FF' = \frac{\sqrt{11}}{2\sqrt{2}}a, \qquad EE' = \sqrt{\frac{11}{2}}a,$$

and the volume of the pyramid is

$$V = \frac{1}{3}\sqrt{\frac{11}{2}}a^3.$$



3. $12^k = 2^{2k}3^k$. Three is a factor in 40 numbers $3n$, $1 \le 3n \le 120$. Because $9 \cdot 13 = 117$, there are 13 numbers $1 \le 9n \le 120$. In addition, there are four numbers $27n < 120$ and one number $81n < 120$. So the maximal k for which 3^k divides 120! is $40 + 13 + 4 + 1 = 58$. In the same way we see that the factor 2 appears in 120! $60 + 30 + 15 + 7 + 3 + 1 = 116 = 2 \cdot 58$ times. So 12^{58} divides 120!, but if $k > 58$, 12^k does not divide 120!.

4. There are just two ways of colouring the squares in the top row so that the white and black squares alternate: "wbwb…" and "bwbw…". Either of these colourings forces the row below to be coloured with alternate colours, but both colourings are possible. It follows that there are 2^n colourings in which the topmost row has this alternating colouring.

Next consider a colouring of the topmost row having at least two adjacent squares of the same colour. If they are e.g. white, the two squares just below have to be black, the squares below these again white etc. In places where the colouring changes, say "bbw", the squares below the black ones have to be white, and this again forces the square below the white square to be black. In general, if three of the squares in a 2×2 square have a definite colour, the fourth square has its colour uniquely determined. So proceeding from two adjacent squares of the same colour, we observe that all squares in the row below have a determined colour, and there again is a pair of adjacent squares of the same colour. This means that the colouring of the top row determines the colouring of the whole $n \times n$ square uniquely. Now there are 2^n possible colourings of the top row, and just two of them have no adjacent squares of the same colour: just the two already considered. So the number of possible colourings is $2^n + 2^n - 2 = 2(2^n - 1)$.

5. If $p = 1$, the problem is trivial. So assume $p < 1$. First of all, it is wiser to answer seven questions instead of eight. If the seventh answer is correct, Mikko passes, and answering correctly to the eighth question does not make the situation any better whereas a wrong answer lowers the score. If the answer to problem seven is wrong, the situation cannot be remedied by an answer to problem eight. A similar argument shows that answering nine questions is better than answering ten questions. So the choice is between seven and nine answers. If Mikko answers seven questions, the probability of passing the test is p and failing $1-p$. If Mikko answers nine questions, he has to get at least two of questions seven, eight and nine right. The probability for this is $p^3+3p^2(1-p)$. We have to find out when

$$p^3 + 3p^2(1-p) > p.$$

The inequality simplifies to

$$p^2 + 3p(1-p) > 1,$$

or

$$(1-p)(3p-1-p) > 0.$$

So the inequality holds for $\frac{1}{2} < p < 1$. So for these p it is more advantageous to answer nine problems. If $p = \frac{1}{2}$, both alternatives are equally good, while in the case $p < \frac{1}{2}$ it is better to answer just seven questions.

Nordic Mathematical Contest 2015

The annual Nordic Mathematics Contest has been running since 1987. It was created to provide potential IMO participants in the five Nordic countries Denmark, Finland, Iceland, Norway and Sweden an experience of a competition slightly more demanding than the national competitions in these countries. It takes place in March or April, and each country can register up to 20 contestants. They all work in their own schools, the papers are first marked in each country and the markings are coordinated by the organizing country, which changes from year to year. The organizing country also compiles the contest paper from suggestions by the participating countries. There are four problems, and the working time is four hours. In 2015, it was Finland's turn to be the organizer. The contest date was March 24.

Problems

1. Let ABC be a triangle and Γ the circle with diameter AB. The bisectors of $\angle BAC$ and $\angle ABC$ intersect Γ (also) at D and E, respectively. The incircle of ABC meets BC and AC at F and G, respectively. Prove that D, E, F and G are collinear.

2. Find the primes p, q, r, given that one of the numbers pqr and $p+q+r$ is 101 times the other.

3. Let $n > 1$ and $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with n real roots (counted with multiplicity). Let the polynomial q be defined by

$$q(x) = \prod_{j=1}^{2015} p(x+j).$$

We know that $p(2015) = 2015$. Prove that q has at least 1970 different roots $r_1, \ldots, r_{1970}$ such that $|r_j| < 2015$ for all $j = 1, \ldots, 1970$.

4. An encyclopedia consists of 2000 numbered volumes. The volumes are stacked in order with number 1 on top and 2000 in the bottom. One may perform two operations with the stack:

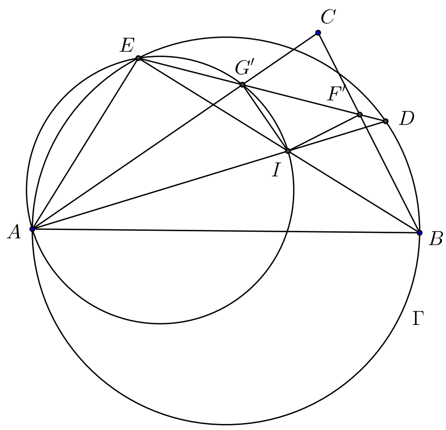
(i) For n even, one may take the top n volumes and put them in the bottom of the stack without changing the order.

(ii) For n odd, one may take the top n volumes, turn the order around and put them on top of the stack again.

How many different permutations of the volumes can be obtained by using these two operations repeatedly?

Solutions

1. Let the line ED meet AC at G' and BC at F'. AD and BE intersect at I, the incenter of ABC. As angles subtending the same arc $\widehat{BD}$, $\angle DAB = \angle DEB = \angle G'EI$. But $\angle DAB = \angle CAD = \angle G'AI$. This means that E, A, I and G' are concyclic, and $\angle AEI = \angle AG'I$ as angles subtending the same chord AI. But AB is a diameter of Γ, and so $\angle AEB = \angle AEI$ is a right angle. So $IG' \perp AC$, or G' is the foot of the perpendicular from I to AC. This implies $G' = G$. In a similar manner we prove that $F' = F$, and the proof is complete.



2. We may assume $r = \max\{p, q, r\}$. Then $p + q + r \leq 3r$ and $pqr \geq 4r$. So the sum of the three primes is always less than their product. The only relevant requirement thus is $pqr = 101(p + q + r)$. We observe that 101 is a prime. So one of p, q, r must be 101. Assume $r = 101$. Then $pq = p + q + 101$. This can be written as $(p - 1)(q - 1) = 102$. Since $102 = 1 \cdot 102 = 2 \cdot 51 = 3 \cdot 34 = 6 \cdot 17$, the possibilities for $\{p, q\}$ are $\{2, 103\}$, $\{3, 52\}$, $\{4, 35\}$, $\{7, 18\}$ The only case, where both the numbers are primes, is $\{2, 103\}$. So the only solution to the problem is $\{p, q, r\} = \{2, 101, 103\}$.

3. Let $h_j(x) = p(x + j)$. Consider h_{2015}. Like p, it has n real roots $s_1, s_2, \ldots, s_n$, and $h_{2015}(0) = p(2015) = 2015$. By Viète, the product $|s_1 s_2 \cdots s_n|$ equals 2015. Since $n \geq 2$, there is at least one s_j such that $|s_j| \leq \sqrt{2015} < \sqrt{2025} = 45$. Denote this s_j by m. Now for all $j = 0, 1, \ldots, 2014$, $h_{2015-j}(m + j) = p(m + j + 2015 - j) = p(m + 2015) = h_{2015}(m) = 0$. So $m, m + 1, \ldots, m + 2014$ are all roots of q. Since $0 \leq |m| < 45$, the condition $|m + j| < 2015$ is satisfied by at least 1970 different j, $0 \leq j \leq 2014$, and we are done.

4. We show by induction that if in an ordered sequence one may exchange two consecutive elements without changing the places of any other element, then any two elements can be exchanged so that all other elements remain in place. We assume that this is true for elements which are at most k steps away from each other in the sequence. Assuming that a precedes b by $k + 1$ steps and that c is immediately behind a, the following sequence of exchanges is allowed: $\ldots, a, c, \ldots, b, \ldots \to \ldots, a, b, \ldots, c, \ldots \to \ldots, b, a, \ldots, c, \ldots \to \ldots, b, c, \ldots, a, \ldots$. By assumption, all elements in the places indicated by three dots remain on their places, as does c.

If any two elements can be exchanged without violating the other elements, then the elements in the sequence can be arranged to any order. One just gets the desired first element to its place by (at most) one exchange, and if the first k elements already are in their desired places, then the one wanted to be in place $k+1$ is not among the first k elements, and it can be moved to its place by at most one exchange, no violating the order of the first k elements.

We now show that any two volumes in consecutive odd places can be exchanged. The volumes on top and in place 3 can be exchanged by operation (ii) applied to the three topmost volumes. The volumes in places $2n+1$ and $2n+3$ can be exchanged by first applying operation (i) to the $2n$ topmost volumes, which moves them in the bottom but preserves their order, then applying (ii) to the three topmost volumes and finally operation (i) to the $2000-2n$ topmost volumes. The last operation returns the $2n$ volumes to top preserving the order and returns the remaining $2000-2n$ volumes to the bottom, preserving the order, save the volumes in places $2n+1$ and $2n+3$, which have changed places. By the general remarks above, it is now clear that operations (i) and (ii) can be used to arrange the volumes in odd positions into any order while the volumes in even positions remain in their places.

We still need to show that a similar procedure is possible for volumes in even positions. First of all, the volumes in positions 1 to 5 can be moved to order 5, 4, 3, 2, 1 by performing operation (ii) to the five topmost volumes. Then it is possible to exchange the volumes in positions 1 and 5 without changing anything else. So the volumes in even positions closest to the top can be exchanged. For volumes on positions $2n$ and $2n+2$ one can first perform operation (i) to the $2n-2$ topmost volumes. The volumes in places $2n$ and $2n+2$ will be taken to places 2 and 4, and they can be exchanged. Performing operation (i) to the $2000-(2n-1)$ topmost volumes then returns everything to their previous places, except that the volumes in positions $2n$ and $2n+2$ have changed places. So all volumes in even positions can be put into any order by using the operations (i) and (ii), and the total number of possible orderings in $(1000!)^2$.