High School Math Contest, Final Round, February 3, 2012

1. A chord divides a circle into two segments. A square is inscribed in each of them in such a way that two vertices of the square are on the chord and two on the circumference. The ratio of the sidelengths of the squares is 5 : 9. Compute the the ratio of the length of the chord to the radius of the circle.

2. Assume $x \neq 1$, $y \neq 1$ and $x \neq y$. Show that if

$$\frac{yz - x^2}{1 - x} = \frac{zx - y^2}{1 - y},$$

then

$$\frac{yz - x^2}{1 - x} = \frac{zx - y^2}{1 - y} = x + y + z.$$

3. Prove that the number $k^{k-1} - 1$ is divisible by the number $(k-1)^2$, for every integer $k \geq 2$.

4. Let k, $n \in \mathbb{N}$, $0 < k \leq n$. Prove that

$$\sum_{j=1}^{k} \binom{n}{j} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \leq n^k.$$

5. The *Collatz function* is a mapping $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that

$$f(x) = \begin{cases} 3x + 1 & \text{for odd } x, \\ x/2 & \text{for even } x. \end{cases}$$

We denote $f^1 = f$ and, inductively, $f^{k+1} = f \circ f^k$, i.e. $f^k(x) = \underbrace{f(\ldots(f}_{k \text{ times}}(x)\ldots)$.

Prove that there is an $x \in \mathbb{Z}_+$ such that

$$f^{40}(x) > 2012x.$$