High School Mathematics Competition 2015

First Round, October 27, Basic Level

The problems are on two pages; the first six problems are multiple choice problems with zero to four correct answers. **The use of calculators is not allowed.**

1. A mouse runs on a conveyor belt from one end of the belt to the other end and back. The speed of the belt u is smaller than the speed v of the mouse. Compared to the situation where the belt does not move, the time the mouse needs for its run

a) is shorter
b) is longer
c) is equal
d) cannot be determined from the given data.

2. What is the smallest real number in the interval $]1, 2[$?

a) 1
b) There is no such number.
c) $1 + 10^{-99}$
d) None of the alternatives a, b or c is correct.

3. A square with side a is divided in two parts by a line parallel to a diagonal of the square. The areas of the parts are in ratio $1 : 4$. The length of the part of the dividing line inside the square is

a) $\dfrac{a}{2}$ b) $\dfrac{a}{\sqrt{2}}$ c) $\dfrac{\sqrt{2}\,a}{\sqrt{5}}$ d) $\dfrac{\sqrt{2}\,a}{2}$

4. A sequence $x_0, x_1, x_2, \ldots$ is defined by setting $x_0 = 2015$ and $x_n = (x_{n-1})^2 + 1$, when n is a positive integer. What can one say of the last digit of the integer x_{2015}?

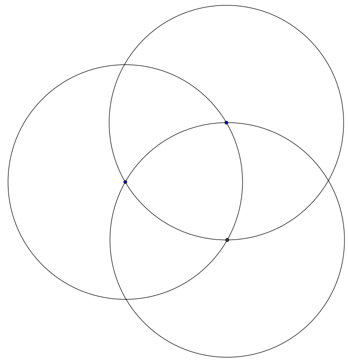
a) It is 2 b) It is 7
c) It is even. d) It is a multiple of 5.

5. The expression $(a + b - c)(a - b + c)(-a + b + c)$ equals for all real numbers a, b and c

a) $\left(a^2 - (b - c)^2\right)(-a + b + c)$
b) $(a + b - c)\left((a - b)^2 - c^2\right)$
c) $-a^3 - b^3 - c^3 + a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - 2abc.$
d) $4\left(ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2 + abc\right) - (a + b + c)^3.$

6. Three circles of radius r are situated so that the centres of each two of the circles lie on the third circle. What can one say of the perimeter p and the area A of the figure?

a) $A < p^2$ b) $p = 3\pi r$
c) $A > 6r^2$ d) $A = (2\pi + \sqrt{3})r^2$



7. Determine all pairs of integers such that their sum is 162 and their greatest common divisor is 18.

8. Prove that $a^{n+4} - a^n$ is divisible by 10, when n and a are positive integers.

High School Mathematics Competition 2015

First Round, October 27, Intermediate Level

Problems

1. A square with side a is divided into two parts by a line parallel to a diagonal of the square. The areas of the parts are in ratio $1:4$. The length of the part of the dividing line inside the square is

a) $\frac{a}{2}$ b) $\frac{a}{\sqrt{2}}$ c) $\frac{\sqrt{2}\,a}{\sqrt{5}}$ d) $\frac{\sqrt{2}\,a}{2}$

2. In how many ways can one write the number 2015 as $p + qrs$, where p, q, r and s are prime numbers and $p < q < r < s$?

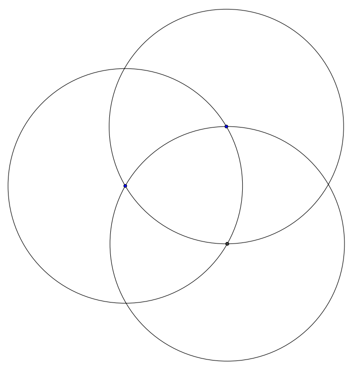
a) In no way. b) In an odd number of ways.
c) In an even number of ways d) In at most ten ways.

3. Let $a,\, b,\, c \in [-1,\, 1]$. What is the largest value of the expression

$$ab + ac + bc + 1 - abc - a - b - c?$$

a) $\frac{1}{2}$ b) 1 c) $\frac{5}{4}$ d) $\frac{3}{2}$

4. Three circles of radius r are situated so that the centres of each two of the circles lie on the third circle. Determine the perimeter p and the area A of the figure thus generated.



5. Let $f : \mathbb{Z} \to \{-1,\, 1\}$ be a mapping which satisfies $f(mn) = f(m)f(n)$ for $m,\, n \in \mathbb{Z}$. Show that there exists an $a \in \mathbb{Z}$ such that $f(a) = f(a+1) = 1$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is *convex*, if for all $a,\, b \in \mathbb{R}$ and $t \in [0,\, 1]$

$$f\left(ta + (1-t)b\right) \leq tf(a) + (1-t)f(b).$$

a) Show that a convex function f satisfies $f\left(ta + (1-t)b\right) + f\left((1-t)a + tb\right) \leq f(a) + f(b)$ for $a,\, b \in \mathbb{R}$, $a < b$, and $t \in [0,\, 1]$.

b) Find out whether the inequality $f(2a-b) \leq 2f(a) - f(b)$ holds for all convex functions $f : \mathbb{R} \to \mathbb{R}$ and all numbers $a,\, b \in \mathbb{R}$, $a < b$.

High School Mathematics Competition 2015

First Round, October 27, Open Level

1. Let a and b be consecutive integers, $c = ab$ and $d = a^2 + b^2 + c^2$.

a) Show that $\sqrt{d}$ is an integer.

b) Can you decide, whether $\sqrt{d}$ is even or odd?

2. The distances of the centre of the incircle of a right triangle from the vertices of the acute angles of the triangle are 2 and 4. Determine the length of the hypothenuse (the exact value).

3. A set A of 41 numbers is given. We know that the sum of any 21 numbers of these numbes is bigger than the sum of the other 20 numbers. What is the maximal number of negative numbers in the set A?

4. Three letters **A**, **B** and **C** are in use. From these one can form e.g. a four letter string **ABBA**. How many different n letter strings with an even number of letters **A** can be formed, when n is a positive integer?

Time allowed: **120 minutes**.

No calculators!

Please write (in legible block letters!) your name and contact information (school, your home address and email address) on your on at least one of the papers you return, and write your name on every paper you return.